Numerical modelling of strain localisation

Panos Papanastasiou\textsuperscript{1} and Antonis Zervos\textsuperscript{2}

\textsuperscript{1}University of Cyprus
\textsuperscript{2}University of Southampton
Introduction

All materials have microstructure!
- Conventional constitutive models ignore this fact.
- But what if microstructure dominates the behaviour?
  - Deformation localization in thin bands.
  - Scale effects.

Photos courtesy of Q. Ni & I. Vardoulakis. Data courtesy of E. Papamichos.
Motivation

Model localisation & scale effect in strain-softening materials

i.e. materials that lose strength as they strain.

- Shearbands in dense sands and overconsolidated clays.
  - Progressive failure of slopes and embankments.
- Localised failure in concrete and rocks.
  - Formation of breakouts in deep boreholes.
- Necking in metals.

...But is that not straight-forward to do with finite elements/differences?

Unfortunately it is far from straight-forward.
Micro-mechanical modelling

- Failure in geomaterials takes place in localized deformation in shear bands
- Modelling localization of deformation requires material softening
- Softening in classical plasticity models results in mesh sensitivity of FE analysis
- Regularized the problem using higher order theories with microstructure, e.g. Cosserat, gradient plasticity, ...
- Internal length: determines the shear band thickness and scale effect
Where the Scale effect is important?

- Mathematical modelling of stability of small holes
- Interpretation of the physical experiments on small holes used to assess the stability of regular (large holes)
  - Elasticity was blamed for failure to predict the hollow cylinder strength
How bad can it be?

Numerical solutions are normally useless
One way out

Mesh sensitivity due to lack of microstructural information.

Use a continuum theory with microstructure:

- Cosserat continuum.
  - Point rotations, as well as displacements.
- Elasticity with Microstructure (Micromorphic Elasticity).
  - Distinct kinematics of micro- and macro-volume.
- Non-local continua.
  - Stress depends on average neighbourhood strain.
- Gradient Elasticity and Gradient Plasticity.
  - Stress depends on strain gradient as well as strain.
Outline

Introduction

Cosserat Continuum

A Gradient Plasticity

Gradient Elastoplasticity

Outlook
Cosserat Continuum
Cosserat continuum

- Independent rotational degrees of freedom
- Non-symmetric stress tensor and couple stresses
- Extended Mohr-Coulomb flow theory of plasticity
- Parameters for Castlegate sandstone, strain softening and internal length related to grain size
Cosserat modelling

components of the relative deformation

\[ \varepsilon_{11} = u_{1,1} \quad \varepsilon_{12} = u_{1,2} + \omega^c \]
\[ \varepsilon_{21} = u_{2,1} - \omega^c \quad \varepsilon_{22} = u_{2,2} \]

two components of curvature

\[ \kappa_1 = \omega_{1}^c \quad \kappa_2 = \omega_{2}^c \]

force and moment equilibrium

\[ \sigma_{ij,j} = 0, \quad m_{k,k} + \sigma_{21} - \sigma_{12} = 0 \quad \text{in} \quad V \]
\[ \sigma_{ij} n_j = t_i, \quad m_i n_i = m \quad \text{on} \quad \partial V \]

incremental strains

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \]
elastic strain and stress increment

\[ d\varepsilon^e_{ij} = \left\{ 2\left( h_1 \delta_{ik} \delta_{j\ell} + h_2 \delta_{jk} \delta_{i\ell} \right) - \frac{k-1}{2k} \delta_{ij} \delta_{k\ell} \right\} \frac{d\sigma_{kl}}{2G} d\kappa^e_i = h_3 \frac{dm_i}{GR^2} \]

Mohr–Coulomb yield criterion

\[ F = \frac{\tau}{p_0 + p} - \mu = 0 \]

\[ \mu = \mu(\gamma^p) \]

Muhlhaus and Vardoulakis (1987)

\[ p = \frac{\sigma_{kk}}{2} \quad \tau = \sqrt{\left( 3s_{ij} s_{ij} - s_{ij} s_{ji} \right) / 4 + m_i m_i / R^2} \]

\[ s_{ij} = \sigma_{ij} + p\delta_{ij} \]

\[ \gamma^p = \int d\gamma^p \quad d\gamma^p = \sqrt{\left( 3d\varepsilon^p_{ij} \varepsilon^p_{ij} + \varepsilon^p_{ij} \varepsilon^p_{ji} \right) / 2 + R^2 d\kappa^p_i d\kappa^p_i} \]

\[ \kappa = K/G = 1/(1-2v) \]
plastic potential

\[ Q = \frac{\tau}{p_0 + p} - \beta \quad \beta = \beta(\gamma^p) \]

flow-rule

\[ d\varepsilon_{ij}^p = d\gamma^p \frac{\partial Q}{\partial \sigma_{ij}} \]

Finite element formulation of Cosserat model

the principle of virtual work

\[ \int_V \{\delta \varepsilon\}^T \{\sigma\} dV = \int_{\partial V} \{\delta u\}^T \{t\} dS \]

\[ \{u\}^T = \{u_1, u_2, \omega^c\} \quad \{\varepsilon\}^T = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{21}, \kappa_1 R, \kappa_2 R\} \]

\[ \{\sigma\}^T = \{\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}, m_1 / R, m_2 / R\} \quad \{t\}^T = \{t_1, t_2, m\} \]
elastic and plastic parts

\[
\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\}
\]

elastic strain and stress increment

\[
\{d\sigma\} = \begin{bmatrix} D^e \end{bmatrix}\{d\varepsilon^e\}
\]

matrix \([D^e]\) contains the elastic parameters of a two dimensional, linear-elastic, isotropic Cosserat continuum defined by

\[
[D^e] =
\begin{bmatrix}
  K + G & K - G & 0 & 0 & 0 & 0 & 0 \\
  K - G & K + G & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & G + G^c & G - G^c & 0 & 0 \\
  0 & 0 & G - G^c & G + G^c & 0 & 0 \\
  0 & 0 & 0 & 0 & M & 0 \\
  0 & 0 & 0 & 0 & 0 & M
\end{bmatrix}
\]
so-called static Cosserat model Muhlhaus and Vardoulakis (1987) proposed

\[
\frac{G^C}{G} = \frac{1}{2}, \quad \frac{M}{G} = R^2
\]

plastic strain and plastic curvature increments

\[
\{d\varepsilon^p\} = d\gamma^p \left\{\frac{\partial Q}{\partial \sigma}\right\}
\]

Prager's consistency condition, \(F = 0\) and \(dF = 0\).

\[
d\gamma^p = \frac{\left\{\frac{\partial F}{\partial \sigma}\right\}\left(\left[D^e\right]\{d\varepsilon\}\right)}{\left\{\frac{\partial F}{\partial \sigma}\right\} \cdot \left(\left[D^e\right]\left\{\frac{\partial Q}{\partial \sigma}\right\}\right) + (p_0 + p) h_t}
\]

plastic modulus

\[
h_t = \frac{d\mu}{d\gamma^p}
\]
stress increment
\[
\{d\sigma\} = [D^{ep}]{d\varepsilon}
\]

stiffness matrix
\[
[D^{ep}] = [D^e] - \langle 1 \rangle \left( [D^e] \left\{ \frac{\partial Q}{\partial \sigma} \right\} \right)^T \left( [D^e] \left\{ \frac{\partial F}{\partial \sigma} \right\} \right) + (p_0 + p) h
\]

Loading of the yield surface \( F = 0 \) takes place when \( d\gamma^p > 0 \)

\[
\langle 1 \rangle = \begin{cases} 
1 & \text{if } F = 0 \text{ and } d\gamma^p > 0 \\
0 & \text{if } F < 0 \text{ or } \{ F = 0 \text{ and } d\gamma^p \leq 0 \}
\end{cases}
\]
Material parameters (hardening-softening)

\[ \mu = \mu_0 + \mu_1 \]

\[ \mu_1 = \frac{(1 - c_0 \gamma^p)\gamma^p}{c_1 + c_2 \gamma^p} \]

\( E = 25 \text{ GPa} \) and Poisson’s ratio, \( \nu = 0.2 \)

**CARBONIFEROUS SANDSTONE**

- \( \mu_0 = 0.175 \)
- \( c_1 = 0.00040767 \)
- \( c_2 = 1.6424 \)
- \( \nu = 0.2 \)
- \( c_0 = 7.5 \)

Mobilized friction coefficient versus accumulated plastic shear strain for a ‘Carboniferous sandstone’.

\( \beta - \mu \) (associated plasticity)
Finite Elements Implementation

\[
\{u\} = [N]\{U\}
\]

\[
\{\Delta \epsilon\} = [B]\{\Delta U\}
\]

\[
[B] = [L][N]
\]

\[
[L] = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & R_{\frac{\partial}{\partial x}} & R_{\frac{\partial}{\partial y}}
\end{bmatrix}
\]

\[
\{\epsilon\} = [L]\{u\}
\]

\[
x(\xi, \eta) = \sum_{i=1}^{8} x_i N_i(\xi, \eta)
\]

\[
y(\xi, \eta) = \sum_{i=1}^{8} y_i N_i(\xi, \eta)
\]

\[
N_1(\xi, \eta) = -0.25(1-\xi)(1-\eta)(1+\xi+\eta)
\]

\[
N_2(\xi, \eta) = 0.5(1-\xi^2)(1-\eta)
\]

\[
N_3(\xi, \eta) = 0.25(1+\xi)(1-\eta)(\xi-\eta-1)
\]

\[
N_4(\xi, \eta) = 0.5(1+\xi)(1-\eta^2)
\]

\[
N_5(\xi, \eta) = 0.25(1+\xi)(1+\eta)(\xi+\eta-1)
\]

\[
N_6(\xi, \eta) = 0.5(1-\xi^2)(1+\eta)
\]

\[
N_7(\xi, \eta) = 0.25(1-\xi)(1+\eta)(-\xi+\eta-1)
\]

\[
N_8(\xi, \eta) = 0.5(1-\xi)(1-\eta^2)
\]
Plasticity integration algorithm

Return mapping

\[ r = \frac{- F(m\{\sigma\}, m\gamma^p)}{\left[ \frac{\partial F}{\partial \{\sigma\}} \right]^T \{\Delta\sigma^e\}} \]

elastic to plastic state
Continuation Methods

Newton-Raphson

Modified N-R

\[ \nabla[B]^T[D_{ep}][\dot{B}]{\Delta U}dV = \lambda_{m+1}\{P\} - \int_\nu[B]^T\{\sigma_m\}dV \]

\[ [K] = \int_\nu[B]^T[D_{ep}][B]dV \]

\[ [K]{\Delta U} = \{R\} \]

\[ \{R\} = \lambda_{m+1}\{P\} - \int_\nu[B]^T\{\sigma_m\}dV \]
Arc-Length Method

Displacement Control

\[
\{ \Delta U \}^{(j)} = \Delta \lambda^{(j)} \{ \Delta U \}^{(i)}_{I} + \{ \Delta U \}^{(j)}_{II}
\]

\[
\{ \Delta U \}^{(j)}_{I} = [K^{(i)}]^{-1} \{ P \}
\]

\[
\{ \Delta U \}^{(j)}_{II} = [K^{(i)}]^{-1} \{ R \}^{(i)}
\]

\[
\lambda_{m+1}^{(j)} = \lambda_{m+1}^{(i)} + \Delta \lambda^{(j)}
\]

Volume control

\[
\Delta V^{(j)} = \Delta \lambda^{(j)} \Delta V^{(j)}_{I} + \Delta V^{(j)}_{II}
\]

\[
\Delta \lambda^{(1)} = \frac{\Delta V - \Delta V^{(1)}_{II}}{\Delta V^{(1)}_{I}}
\]

\[
\Delta \lambda^{(j)} = -\frac{\Delta V^{(j)}_{II}}{\Delta V^{(j)}_{I}}
\]
Softening in classical plasticity
Borehole Failure

- failure in geomaterials takes place in localized deformation in shear bands
Borehole analysis
Bifurcation analysis

conditions for a bifurcation:

\[ \int_V \Delta \sigma_{ij} \Delta \varepsilon_{ij} dV = 0 \]

\[ \det([K]) = 0 \]
Eigen-value analysis

<table>
<thead>
<tr>
<th>real part</th>
<th>imaginary part</th>
<th>eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.51896E-03</td>
<td>0.00000E+00</td>
<td>10</td>
</tr>
<tr>
<td>0.80039E-03</td>
<td>0.00000E+00</td>
<td>12</td>
</tr>
<tr>
<td>0.14811E-02</td>
<td>0.00000E+00</td>
<td>8</td>
</tr>
<tr>
<td>0.21925E-02</td>
<td>0.00000E+00</td>
<td>14</td>
</tr>
<tr>
<td>0.25344E-02</td>
<td>0.00000E+00</td>
<td>16</td>
</tr>
</tbody>
</table>

Detail of eigenvector with element state at first step of localization, warping mode $m = 12$ ($r_0 = 6$ cm, $\sigma_{d}/p_0 = 2.388$).

Papanastasiou and Vardoulakis (1992)
Isotropic stress-field
$s_h/S_h = 1.001$

$r_o = 3 \text{ cm}$
Anisotropic stress-field

Fig. 13. Global picture of elastic-plastic domains (a) before localization, (b) after localization. Details of (c) elastic-plastic domains, (d) isolines of accumulated plastic shear strain, $\gamma^p$, (e) incremental displacement field, (f) deformed mesh, after localization ($r_0 = 3 \text{ cm, } S_0/P_0 = 2.053, S_1/S_0 = 1.5$).
Breakouts prediction

<table>
<thead>
<tr>
<th>$r_o$, cm</th>
<th>$S_H/S_h$</th>
<th>$\theta^o_b$</th>
<th>$r_b/r_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.0</td>
<td>58</td>
<td>1.42</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>32</td>
<td>1.25</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td>30</td>
<td>1.23</td>
</tr>
<tr>
<td>6</td>
<td>3.0</td>
<td>22</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Fig. 14. Comparison of experimental and computational results.

Experimental results
Haimson and Herrick, 1989
Comparison with thick-walled cylinder test

Papanastasiou and Zervos, 2000
Elliptical Shape Perforations: an engineering application based on Cosserat modelling

- Introduction
  - sand avoidance from perforated intervals
- Computational results
  - elastic analysis
  - Cosserat elastoplastic analysis
- Conclusion
  - practical application
Sand production and avoidance

- **Sanding problem** ($2 billion/year)
  - blocks perforations, damages of equipment, requires separation from the oil and disposal

- **Avoidance**
  - gravel packing and screening, preferential perforating, fracturing

- **Objective**
  - develop models to predict sanding and optimum completion

Real perforation
Mechanisms of sand production

- Hollow Cylinder Tests
  - various weak sandstones
  - 10-20 mm perforation size
  - hydrostatic pressure

- Sand production in two stages
  - stresses due to drawdown and depletion fail the rock
  - high flow velocities transport loose grain

- Unconsolidated formation
  - erosion mechanism

Cook and Nicholson (SCR)
Production from perforated intervals

Perforating tool

Real perforation
Sand avoidance

- Perforation failure caused by high compressive stresses results in sand production.
- Redistribute the stresses on perforations by changing their shape:
  - elliptically shaped perforations
Elastic stress analysis

- uniform stress distribution if axis ratio is equal to the insitu stress ratio
- risk of perforation misalignment 23 degrees
compare cylindrical with elliptical perforations of the same cross-sectional area (flow area)

strongest perforations: ellipse with the highest axis ratio
Conclusion

- elliptically shaped perforations are stronger than conventional perforations
- this result was found using Cosserat modelling
- results predicted by classical stress analysis were not applicable
- Cosserat allows for robust localization analysis
- Limited applications in design and practise
A Gradient Plasticity
Motivation

There are many gradient plasticity theories in the literature:

- Vardoulakis, Aifantis, de Borst and Pamin, Fleck and Hutchinson, Chambon, Zbib... (The list is non-exhaustive.)

It is impossible to review them all here.

- We will focus on numerical computations.
Biaxial test revisited

(de Borst and Pamin, 1996)

But:

- Equations change order at the elastoplastic boundary (this is inconvenient).
- The way of introducing the internal length is perhaps counter-intuitive.
Gradient Elastoplasticity

Motivation
Constitutive relations
Internal stress-like quantities
Finite element formulation
Biaxial test
Thick-cylinder test
Cavity expansion

Outlook
Motivation

Model deformation and failure of geomaterials.

- Significant irreversible (plastic) deformation.
- Strain softening leads to deformation localisation.
- Existence of scale effects.

Introduce microstructure in plasticity.

- Build on the ideas of gradient elasticity.
  - Governing equations do not change order.
  - No boundary conditions on elastoplastic boundary.
Gradient elastoplasticity

Total (equilibrium) stress rate
\[ \dot{\sigma}_{ij} = C_{ijkl}^e (\dot{\epsilon}_{kl}^e - l_c^2 \nabla^2 \epsilon_{kl}^e) \]

Yield function and plastic potential
\[ F(\tau_{ij}, \psi) = 0, \quad Q(\tau_{ij}, \psi) = 0 \]

Plastic strain rate
\[ \dot{\epsilon}_{ij}^p = \psi \frac{\partial Q}{\partial \tau_{ij}} \]

Reduced stress rate
\[ \dot{\tau}_{ij} = \dot{\sigma}_{ij} - \dot{\alpha}_{ij} \]

Back stress rate
\[ \dot{\alpha}_{ij} = -l_p^2 C_{ijkl}^e \nabla^2 \epsilon_{kl}^p \]
Finite element formulation

- Use a displacement formulation: \( \mathbf{u} = \mathbf{N} \cdot \mathbf{u} \)
- Strain gradient in the weak form: \( C^1 \)-continuous element.

\( C^1 \) triangle.

- 36 degrees of freedom.
- Quintic polynomial.
- Cubic normal derivative.
Modelling the biaxial test

The different meshes used:

<table>
<thead>
<tr>
<th>Name</th>
<th>Mesh</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse</td>
<td>10x20</td>
<td>2772</td>
</tr>
<tr>
<td>Medium</td>
<td>20x40</td>
<td>10332</td>
</tr>
<tr>
<td>Fine</td>
<td>30x60</td>
<td>22692</td>
</tr>
<tr>
<td>Very fine</td>
<td>40x80</td>
<td>39852</td>
</tr>
</tbody>
</table>

A. Zervos & P. Papanastasiou

ALERT Graduate School, Aussois, 2010 – 21 /39
Intr

doction

A Gradient Plasticity

Gradient
Elastoplasticity

Motivation
Constitutive
relations

Internal stress-like
quantities

Finite element
formulation

Biaxial test

Thick-cylinder test

Cavity expansion

Outlook

Displ. increment, plastic strain and material state. \( \phi = \psi = 32^0 \)
Contours of plastic strain for different dilatancy angles.

- Simulations capture quantitatively the failure mechanism:
  - Shearband inclination: $\theta \approx 45^\circ + (\varphi + \psi)/4$
  - Reorientation near a free boundary: $\theta \approx 45^\circ + \psi/2$

Photo courtesy of I. Vardoulakis
Modelling the thick-cylinder test

Stability of wellbores and perforations.
External pressure increased to failure.

The different meshes used:

<table>
<thead>
<tr>
<th>Name</th>
<th>Mesh</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse</td>
<td>15x40</td>
<td>7380</td>
</tr>
<tr>
<td>Medium</td>
<td>20x80</td>
<td>19440</td>
</tr>
<tr>
<td>Fine</td>
<td>25x121</td>
<td>36300</td>
</tr>
</tbody>
</table>

\[ R_{\text{ext}} = 5 R_{\text{int}} \]
Loss of symmetry and final breakout mechanism.
Introduction

A Gradient Plasticity

Gradient Elasticity

Motivation

Constitutive relations

Internal stress-like quantities

Finite element formulation

Biaxial test

Thick-cylinder test

Cavity expansion

Outlook

A. Zervos & P. Papanastasiou

ALERT Graduate School, Aussois, 2010 – 26 /39
Introduction

Gradient Plasticity
Gradient Elastoplasticity
Motivation
Constitutive relations
Internal stress-like quantities
Finite element formulation
Biaxial test
Thick-cylinder test
Cavity expansion

Outlook

---

**Photo courtesy of A. Guenot; scale effect data courtesy of E. Papamichos.**
Post-Bifurcation Analysis

Finite element formulation

- **Displacements:** \( u = N \cdot \hat{u} \)
- **\( u \) must be \( C^1 \) continuous**
- **We use a \( C^1 \) triangle (36 DOFs)**
For the case of $R_i = 10$ cm and a mesh with 32640 DOFs

- Spontaneous loss of axisymmetry, $m = 30$
- Deformation localisation in thin, softening bands
For the case of $R_i = 10$ cm and a mesh with 32640 DOFs

- Spontaneous loss of axisymmetry, $m = 30$
- Deformation localisation in thin, softening bands
- The maximum pressure is lower than the limit pressure
For the case of $R_i = 10 \text{ cm}$ and a mesh with 32640 DOFs

- Spontaneous loss of axisymmetry, $m = 30$
- Deformation localisation in thin, softening bands
- The maximum pressure is lower than the limit pressure
Outlook
Outlook

Higher order theories:

- Introduce material length scales.
- Regularise material behaviour in the softening regime.
- Are able to capture localised failure mechanisms.
  - Finite shearband thickness.
  - Robust numerical solutions post-peak.
  - Robust predictions and tracking of instabilities.
- Are able to capture scale effects.

However there are still issues to be resolved...
...and some of these issues are:

- New parameters: material lengths and softening rate.
- How do we calibrate?
- Element tests are NOT the answer!
...and some of these issues are:

- New parameters: material lengths and softening rate.
  - How do we calibrate?
  - Element tests are NOT the answer!

- New types of boundary conditions: richer behaviour.
  - Physical meaning not always clear.
  - Restriction of derivatives is analogous to “roughness”.

---

A. Zervos & P. Papanastasiou
ALERT Graduate School, Aussois, 2010 – 38 / 39